The limits of determinacy in third-order arithmetic

Juan P. Aguilera and Thibaut Kouptchinsky[†]

[†] Research Unit of Computational Logic, TU Wien

TECHNISCHE UNIVERSITÄT WIEN

A Striking Result of Montalbán and Shore

We wish to extend the results of [2] to third-order arithmetic. A central question in Reverse Mathematics is the characterisation of the proof-theoretic strength of theorems of mathematics. Determinacy axioms are on their side powerful statements whose proof-theoretic strength grows fast.
Theorems 1 & 2 [2] : For every $n \geq 1$,

$$\Delta^1_{n+2}\text{-}\mathsf{CA}_0 \not\geq (\Pi^0_3)_n\text{-}\mathsf{Det} < \Pi^1_{n+2}\text{-}\mathsf{CA}_0.$$

A model of Γ-CA_0 is a structure $\mathcal{M} = (M, S, +, \times, <, 0, 1, \in)$, with $(M, +, \times, <, 0, 1)$ being an ordered semi-ring, M a set of numbers and $S \subseteq \mathcal{P}(M)$ the collection of subsets of numbers, that satisfy the axiom of Γ comprehension and Γ induction. The backbone of the zoo of Reverse Mathematics is

$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0 < \cdots < \Pi^1_{n+1}\text{-}\mathsf{CA}_0 < \cdots < \mathsf{Z}_2.$$

To have a set-theoretic equivalent in mind, in L, the minimal model of Δ^1_{n+2}-CA_0 has the same set S than $L_{\alpha^1_{n+1}}$, the minimal level α so that

$$L_\alpha \models \mathsf{KP}_{n+1} \coloneqq \mathsf{KP} + \Delta_n\text{-}\textsc{collection} + \Sigma_n\text{-}\textsc{separation} \qquad (n \geq 1).$$

Determinacy Axioms

Two players, I and II give natural numbers one after the other for an infinite amount of time. Once each move has been stated, it becomes common knowledge, making it a perfect information game. The legal moves are represented by the tree T, containing all legal finite positions of the games. Player I wins iff the sequence of their alternative moves belongs to $X \subseteq [T]$, otherwise player II wins.

A: a_0 $\quad a_2 \quad\quad a_{2n}$ $\quad \cdots \quad \cdots \quad (a_i)_{i<\omega} \overset{?}{\in} X$

B: $\quad a_1 \quad a_3 \quad\quad a_{2n+1}$

Legal move condition: $\forall n\ (a_0, a_1, \ldots, a_n) \in T \subseteq \mathbb{N}^{<\omega}$

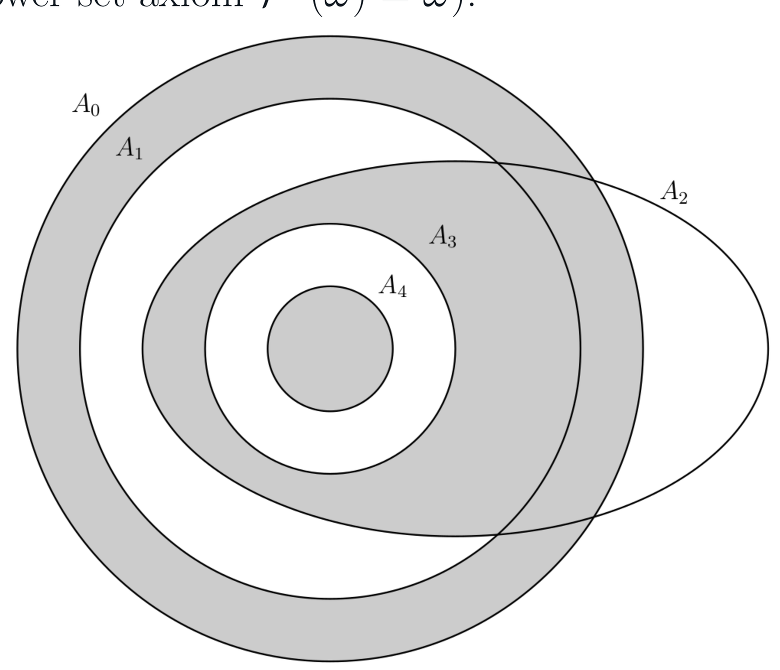
A strategy σ is **winning** for player I (resp. II) if $[\sigma] \subseteq X$ (resp. $[\sigma] \subseteq \bar{X}$). A game $G(T, X)$ is said to be **determined** if there exists a strategy for one of the two players. Using the natural topology on $[T]$, one can states the axiom of determinacy restricted to a class Γ of definable sets as Γ-Det^T: $\forall X \in \Gamma,\ G(T, X)$ is determined. A theorem of Martin [1] states for all $\alpha < \omega_1$

$$\mathsf{ZFC}^- + \text{“}\mathcal{P}^\alpha(\omega) \text{ exists”} \not\geq \Pi^0_{\alpha+4}\text{-}\mathsf{Det} \leq \mathsf{ZFC}^- + \text{“}\mathcal{P}^{\alpha+1}(\omega) \text{ exists”},$$

(with ZFC^- is ZFC deprived from power set axiom $\mathcal{P}^0(\omega) = \omega$).

We define the n-th difference hierarchy level $(\Pi^0_3)_n$ by the following construction on the n-th Π^0_3 sets,

$$(A_0 \backslash (A_1 \setminus (A_2 \setminus \cdots \cdots (A_{n-2} \setminus A_{n-1}) \cdots))).$$



Results

The goal of the present paper is to extend the result of Montalbán and Shore in **third-order arithmetic**. In future work, we plan to address questions of tightening the bounds as well as the limits of real determinacy in third-order arithmetic. Let us denote by α^2_n the least ordinal such that $L_{\alpha^2_n} \models \mathsf{KP}_n + \mathcal{P}(\omega)$ exists. Our result is the following:

$$L_{\alpha^2_n} \not\models (\Pi^0_4)_n\text{-}\mathsf{Det} \quad \text{but} \quad L_{\alpha^2_{n+1}} \models (\Pi^0_4)_n\text{-}\mathsf{Det} \qquad (n \geq 2),$$

where within **countable** case, we had

$$L_{\alpha^1_{n+1}} \not\models (\Pi^0_3)_n\text{-}\mathsf{Det} \quad \text{but} \quad L_{\alpha^1_{n+2}} \models (\Pi^0_3)_n\text{-}\mathsf{Det} \qquad (n \geq 1).$$

The way we prove the first result is by devising a $(\Pi^0_4)_n$ game such that

1. If player I plays the theory of $L_{\alpha^2_n}$, she wins;
2. If player I does not play the theory of $L_{\alpha^2_n}$ but player II does, then player II wins.

	winning condition for II	winning condition for I
A_0	$\neg C_{\mathrm{I}}0 \vee [C_{\mathrm{II}}0 \wedge C_{\mathrm{II}}1]$	
A_1		$C_{\mathrm{I}}0 \wedge C_{\mathrm{I}}1 \wedge C_{\mathrm{I}}2$
	$\vdots$	$\vdots$
A_{2j}	$(C_{\mathrm{II}}1 + (2j-1)) \wedge (C_{\mathrm{II}}(1+2j))$	
A_{2j+1}		$(C_{\mathrm{I}}(1+2j)) \wedge (C_{\mathrm{I}}1 + (2j+1))$
	$\vdots$	$\vdots$
A_{n-1}	$(C_{\mathrm{II}}n)$	

With each played theory we associate his term model $\mathcal{M}_{::}$. The conditions

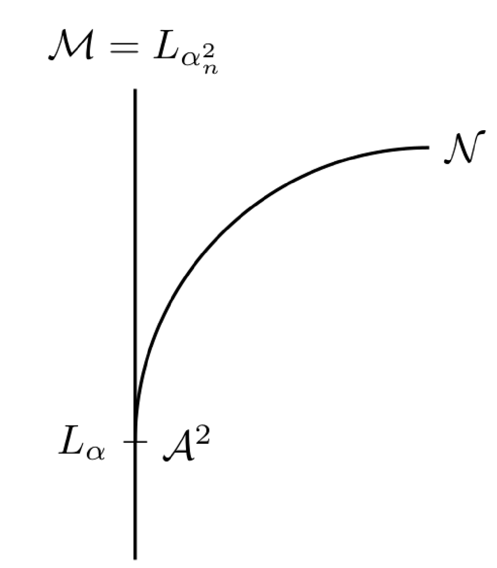
- $C_{::}0$ check if $\mathcal{M}_{::}$ is a minimal omega model of the theory not included in the other;
- $C_{::}1$ check that there is no descending sequence of countable non-standard codes;
- $C_{::}(1+k)$ check that an induction hypothesis $\star_{k-1}$ is witnessed and there is no $\Delta_0(\Sigma_{k-1})$ descending sequence in $\mathcal{M}_{::}$.

Not all rules can be satisfied since the existence of $\star_{k-1}$-witnesses implies that $\mathcal{M}_{::}$ has a strict sub-model of $\mathsf{KP}_k + \mathcal{P}(\omega)$ exists. So the satisfaction of $C_{::}[0:n]$ would contradict $C_{::}0$.

The Peculiarity of the Uncountable Case

Crucially in the statement of the condition $C_{::}(1+k)$, we need to identify the well-founded, common part of the two models played. A major hindrance in generalising the definition of Montalbán and Shore occurs from the possible positions of ω_1, the highest cardinal in both models. They are three cases.

1. Both ω_1 belongs to L_α and the generalisation goes smoothly;
2. Only one of the ω_1 is out of L_α, then when comparing the ordinals, we keep looking at ordinals under the least countable non-standard code when considering the ones of the model to whom belong this ω_1;
3. Both ω_1 are out of L_α, then for both model, we only compare ordinals under the least countable non-standard code of each model.



A typical situation in the game of KP^2_n

Moreover, by overspill, the extra condition imply that L_α is a model of $\mathcal{P}(\omega)$ exists.

Remarks and Exact Bounds

The second part of our result is an application of Martin's unraveling, combined with a rather immediate adaptation of Martin's original proof of the determinacy of the hierarchy of differences of Π^0_3 sets. By doing so, one of the original quantifiers become bounded, which explain that less comprehension is needed. Our results can be generalised to any level of the Borel hierarchy. By extending our method to the second paper of Montalbán and Shore [3], we were able to answer to a question of Pachecho and Yokoyama [4]. We settled for all $i < \omega$

1. $\forall n\ (\Pi^0_{3+i})_{(n+1)}$-$\mathsf{Det}$ is equivalent to Π^1_3-$\mathsf{Ref}(\mathsf{ZFC}^i)$;
2. $\forall m\ (\Pi^0_m)$-Det is equivalent to Π^1_3-$\mathsf{Ref}(\{\mathsf{ZFC}^m\}_{m<\omega})$.

References

[1] Donald A. Martin. *Determinacy of Infinitely Long Games*. https://www.math.ucla.edu/~dam/booketc/D.A._Martin,_Determinacy_of_Infinitely_Long_Games.pdf.

[2] Antonio Montalbán & Richard A. Shore. "The limits of determinacy in second order arithmetic". In: *Proceedings of the London Mathematical Society* 104.2 (), pp. 223–252.

[3] Antonio Montalbán & Richard A. Shore. "The limits of determinacy in second order arithmetic: consistency and complexity strength". In: *Israel Journal of Mathematics* 204 (), pp. 477–508.

[4] Leonardo Pachecho & Keita Yokoyama. "Determinacy and reflection principles in second-order arithmetic". In: *arXiv* ().